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Abstract. We introduce two new characterizations of Meyer sets. A repetitive Delone set in $\mathbb{R}^d$ with finite local complexity is topologically conjugate to a Meyer set if and only if it has d linearly independent topological eigenvalues, which is if and only if it is topologically conjugate to a bundle over a d-torus with totally disconnected compact fiber and expansive canonical action. "Conjugate to" is a non-trivial condition, as we show that there exist sets that are topologically conjugate to Meyer sets but are not themselves Meyer. We also exhibit a diffractive set that is not Meyer, answering in the negative a question posed by Lagarias, and exhibit a Meyer set for which the measurable and topological eigenvalues are different.

1. Introduction

To provide a rigorous mathematical explanation of the observation that certain non periodic media (quasicrystals) show sharp Bragg peaks in their X-ray diffraction, mathematicians came up with the notion of a *pure point diffractive* set. This is a point set Λ of $\mathbb{R}^d$ to which can be associated an auto-correlation measure γ whose Fourier transform $\hat{\gamma}$ is a pure point measure [Hof] (see also [Mo2] and references therein). In the model of the material by the point set, the Bragg peaks correspond exactly to the points in $\mathbb{R}^{d*}$ that have strictly positive $\hat{\gamma}$-measure. This raises the question of which point sets are pure point diffractive. Whereas an answer in terms of the properties of the autocorrelation measure γ has been found, namely that this is the case whenever γ is strongly almost periodic [BM, G, LA] no complete geometric characterization of such point sets is known.

The question can be reformulated as a property of the measurable dynamical system $(\Omega, \mathbb{R}^d, \mu)$. Here Ω is the hull of Λ, a compact metrizable space consisting of points sets whose patches look like those of Λ and on which $\mathbb{R}^d$ acts by translation, and μ is an invariant ergodic Borel probability measure that is determined by the patch frequencies. A point $\beta \in \mathbb{R}^{d*}$ is called an eigenvalue (or dynamical eigenvalue) for Λ if there exists an L^2-function f (an eigenfunction) which satisfies the eigenvalue equation

$$f(\omega - t) = e^{2\pi i \beta(t)} f(\omega) \tag{1}$$

for μ-almost all $\omega \in \Omega$ and all $t \in \mathbb{R}^d$. Arguments based on work of Dworkin [Dw, LMS] showed that any Bragg peak gives rise to a dynamical eigenvalue. Moreover, Λ is pure point

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diffractive whenever the measurable dynamical system $(\Omega, \mathbb{R}^d, \mu)$ has pure point dynamical spectrum, that is, whenever $L^2(\Omega, \mu)$ is spanned by eigenfunctions.

Interestingly enough, there are classes of point sets for which pure point diffractivity becomes a property of *topological* dynamical systems. An eigenvalue β is called a continuous or *topological* eigenvalue if (1) has a *continuous* solution. When the point set Λ comes from a substitution it is known that that all eigenvalues are topological [H, So]. Furthermore, repetitive regular model sets, which are known to be pure point diffractive, also have only topological eigenvalues [Hof, Sch].

However, there do exist diffractive point sets whose measurable eigenvalues are not all topological (see Section 8). It then makes sense to ask about the topological eigenvalues. For which point sets are all the eigenvalues continuous?

An experimental material scientist may doubt the usefulness of the mathematical concept of pure point diffraction for X-ray analysis, since experimental devices only have finite resolution. Since we can only see Bragg peaks above a given brightness s, we can never observe a dense set of peaks. At best, we can observe a *relatively* dense set of peaks, meaning there is a radius r such that every ball of radius r in $\mathbb{R}^{d*}$ contains at least one peak.

Recently, Nicolae Strungaru [St1, St2] showed that Meyer sets have this property for any s below the maximal intensity of the Bragg peaks. In other words, if you can see at least one Bragg peak from a Meyer set Λ, and if you increase the sensitivity of your equipment even slightly, then you will see a relatively dense set of Bragg peaks. From this point of view all Meyer sets should be regarded as being diffractive, but not in the sense of having *pure* point diffraction spectrum. Instead, a point set with relatively dense set of Bragg peaks is called *essentially diffractive* [St2].

Dworkin's argument now yields a necessary criterion of essential diffractivity: Λ must have d linearly independent (measurable) eigenvalues. We then ask: is this criterion a topological property for certain classes of point sets? In other words, which point sets have d linearly independent *topological* eigenvalues? One of our main results gives a characterization of such sets:

Theorem 1.1. *A repetitive Delone set with finite local complexity has d linearly independent topological eigenvalues if and only if it is topologically conjugate to a Meyer set. In particular, each repetitive Meyer set has d linearly independent topological eigenvalues.*

Our theorem can be combined with a characterization of regular model sets given in [BLM] to obtain:

Theorem 1.2. *A repetitive Delone set with finite local complexity is topologically conjugate to a repetitive regular model set if and only if its dynamical system is uniquely ergodic and the continuous eigenfunctions separate almost all points of its hull.*

Our analysis draws on the use of pattern equivariant functions and of certain fiber bundles. Theorem 1.1 is actually proven via a third characterization of Meyer sets, which may well have its own benefits:

Theorem 1.3. *The dynamical system of a repetitive Delone set of finite local complexity is topologically conjugate to a bundle over the d-torus with totally disconnected compact fiber and expansive canonical $\mathbb{R}^d$-action if and only if it is topologically conjugate to the dynamical system of a Meyer set.*

If we add the requirement that the Delone set is not fully periodic then the fiber of the bundle is a Cantor set.

The above theorems characterize Delone sets up to topological conjugacy, by which we mean topological conjugacy of their associated dynamical system. To obtain a geometric characterization one needs to understand to which extend the dynamical system of a Delone set determines the Delone set and which geometric propeties of a Delone set are preserved under topological conjugacy. For this the following theorem is useful.

Theorem 1.4. *Any topological conjucacy between the dynamical systems of Delone sets of finite local complexity is the composition of a shape conjugation with a mutual local derivation. Furthermore, the shape conjugation can be chosen to move points by an arbitrarily small amount.*

A natural question is whether a Delone set of finite local complexity (FLC) that is shape conjugate to a Meyer set must itself be a Meyer set. As a consequence of the above theorem, any FLC Delone set that is shape conjugate to a Meyer set is arbitrarily close in the Hausdorff distance to a Meyer set. However, this does not suffice to guarantee that it is Meyer. Indeed, we provide a counterexample. The Meyer propety is invariant under local derivations but not under shape conjugacy.

It is not surprising that Meyer sets show up as solutions to the topological version of essential diffractivity. They have been investigated throughout the efforts to describe sets which are pure point diffractive. Lagarias (see [La2], Problem 4.10) suggested the problem of proving that a pure point diffractive set is necessarily Meyer. This turns out to be false. A recent example of [FS], called the scrambled Fibonacci tiling, provides a counterexample. It has pure point dynamical spectrum, but none of the eigenfunctions can be chosen continuous, and hence it is not a Meyer tiling (something that can also be checked directly).

A variation of the scrambled Fibonacci yields a Meyer set that has dense pure point dynamical spectrum but only relatively dense topological dynamical spectrum. The dynamical spectrum has rank 2 while the topological spectrum only has rank 1.

2. Preliminaries

We assume that the reader is familiar with the standard objects and concepts from the theory of tilings or Delone sets, in particular with the notions of finite local complexity (FLC) and repetitivity. We also suppose that the construction of the hull Ω_Λ and the dynamical system $(\Omega_\Lambda, \mathbb{R}^d)$ associated with a tiling or a Delone set $\Lambda \subset \mathbb{R}^d$ is known. Where necessary we assume the existence of frequencies of patches defined by a van Hove sequence and hence of an ergodic probability measure μ on the hull, see, for instance, [So].

An important well-known concept for us is that of a local derivation (or local map) between two point sets or tilings. A point set Λ' is locally derivable from a point set Λ if there exists $R > 0$ such that the question of whether a point x belongs to Λ' can be answered by inspection of $B_R(0) \cap (\Lambda - x)$ [BSJ]. This defines a map between the hulls Ω_Λ and $\Omega_{\Lambda'}$ which is called a local derivation. The definition applies literally to tilings if we identify the tiling with a subset of $\mathbb{R}^d$ of boundary points of its tiles. There are many ways to convert Delone sets to tilings or to convert tilings to Delone sets in a mutually locally derivable way, i.e. with bijective maps with are in both directions local derivations. We simply say then that the two objects are MLD. Given a (polyhedral) FLC tiling $\mathbf{T}$, the set of its vertices is an FLC Delone set which is locally derivable from the tiling. Given an FLC Delone set Λ, the collection of Voronoi cells is an FLC tiling, and the vertices of the dual to this Voronoi tiling (the so-called Delone tiling) are precisely the elements of Λ. We denote by $\mathbf{T}_\Lambda$ the Delone tiling of Λ which is clearly mutually locally derivable with Λ. This paper is written primarily in the language of Delone sets, but we freely use theorems about tilings.

When we speak about eigenfunctions and eigenvalues for a Delone set Λ (or a tiling), what we mean are eigenfunctions for the system $(\Omega_\Lambda, \mathbb{R}^d, \mu)$, and when we say that two Delone sets Λ and Λ' are topologically conjugate we mean that their dynamical systems $(\Omega_\Lambda, \mathbb{R}^d)$ and $(\Omega_{\Lambda'}, \mathbb{R}^d)$ are topologically conjugate.

3. Meyer sets

A subset $\Lambda \subset \mathbb{R}^d$ is called *harmonious* if any algebraic character on the group $\langle \Lambda \rangle$ generated by Λ can be arbitrarily well approximated by a continuous character. Continuity refers here to the relative topology of $\langle \Lambda \rangle \subset \mathbb{R}^d$ and so a continuous character is of the form $e^{2\pi i \beta}$ for some $\beta \in \mathbb{R}^{d^*}$. Very roughly speaking, harmonious sets are (potentially) non-periodic sets to which harmonic analysis can still be applied. Harmonious play an important role in Meyer's book [Me1]. In particular the class of harmonious sets which Meyer called *model sets* (see [Mo1]) plays an important role in spectral synthesis [Me1] and in optimal and universal sampling in information theory [Me1, MM, Me3]. After the discovery of quasicrystals, the relevance of harmonious sets and model sets to the diffraction was recognized [Me2, Mo1, La1] and nowadays a relatively dense harmonious set is called a *Meyer set*. A tiling is a Meyer tiling if it is MLD to a Meyer set.

Harmonious sets allow for various different characterizations. Some of them are analytical but there is also a surprisingly simple geometric criterion. Let

$$\Lambda^\epsilon = \{\beta \in \mathbb{R}^{d*} | \forall a \in \Lambda, |1 - \exp(2\pi i \beta(a))| < \epsilon\}.$$

The following conditions are equivalent [Me1, Mo1, La1].

(1) Λ is harmonious, that is, for any group homomorphism $\phi : \langle \Lambda \rangle \to S^1$ and any $\epsilon > 0$, there is a $\beta \in \mathbb{R}^{d*}$ such that $|\phi(a) - \exp(2\pi i \beta(a))| < \epsilon$ for all $a \in \Lambda$.
(2) Λ^ϵ is relatively dense for all $\epsilon > 0$.
(3) Λ^ϵ is relatively dense for some $0 < \epsilon < 1/2$.
(4) $\Lambda - \Lambda$ is uniformly discrete.
(5) Λ is uniformly discrete and there exists a finite set F such that $\Lambda - \Lambda = \Lambda + F$.
(6) Λ is a subset of a model set.

Note that conditions (4) and (5) are manifestly preserved by MLD transformations, so a set that is MLD to a Meyer set is necessarily Meyer.

The two characterizations we will use here are 3 and 4. It follows from 4 that a Meyer set is an FLC Delone set. Note that $\Lambda - \Lambda$ is always discrete if Λ is FLC, but the condition that $\Lambda - \Lambda$ is *uniformly* discrete is very strong: it implies, for instance, that $\Delta = \Lambda - \Lambda$ is also a Meyer set [Mo1]. Hence Δ^ϵ is relatively dense for all $\epsilon > 0$; that is, for a relatively dense set of β's we can then find a plane wave $e^{2\pi i \beta}$ that has nearly the same phase at all points in Λ. This will be the key to constructing continuous eigenfunctions.

Model sets are point sets obtained from a cut & project scheme [Mo1] with very strong properties which makes them suitable for the description of quasicrystals. As to our knowledge, no direct geometric characterization of model sets is known. Using their associated dynamical system they can be characterized in the following way:

Theorem 3.1 ([BLM]). $(\Omega, \mathbb{R}^d)$ *is the dynamical system of a repetitive regular model set, if and only if*

(1) *all elements of* Ω *are Meyer sets,*
(2) $(\Omega, \mathbb{R}^d)$ *is minimal and uniquely ergodic,*
(3) $(\Omega, \mathbb{R}^d)$ *has pure point dynamical spectrum and all eigenvalues are topological, and*
(4) *the continuous eigenfunctions separate almost all points of* Ω.

Since the maximal equicontinuous factor of a dynamical system is the spectrum of the algebra generated by its continuous eigenfunctions, the last property is equivalent to the fact that the factor map $\pi_{max} : \Omega \to \Omega_{max}$ from Ω to its maximal equicontinuous factor Ω_{max} is almost everywhere 1:1, and hence induces a measure isomorphism between $L^2(\Omega, \mu)$ and $L^2(\Omega_{max}, \eta)$ (η is the Haar measure on Ω_{max}) (see e.g. [BK]). Taking into account that $C(\Omega_{max})$ is dense in $L^2(\Omega_{max}, \eta)$ it follows that Property 4 implies Property 3. Moreover, by Theorem 1.1 Property 3 implies that $(\Omega, \mathbb{R}^d)$ is topologically conjugate to the dynamical system of a Meyer set and hence the above theorem simplifies to Theorem 1.2.

4. Pattern equivariant functions and cochains

We recall the definitions of pattern equivariant functions and cochains for uniformly discrete FLC point sets or FLC tilings [K1, Sa].

Let Λ be a uniformly discrete point set or tiling in $\mathbb{R}^d$ of finite local complexity. We call a function $f : \mathbb{R}^d \to \mathbb{C}$ (or into any abelian group A) strongly pattern equivariant if there exists a radius R such that $f(x) = f(y)$ for any two points x, y with $(\Lambda - x) \cap B_R(0) = (\Lambda - y) \cap B_R(0)$. In other words, for each x, $f(x)$ is determined by the pattern of points within a fixed finite distance R of x. We can also consider a tiling as a decomposition of $\mathbb{R}^d$ into 0-cells, 1-cells, $\ldots$, and d-cells. A k-cochain assigns a number to each k-cell. Such a cochain is called strongly pattern equivariant if the value of a k-cell depends only on the pattern of tiles within a distance R of that k-cell.

An equivalent definition of pattern equivariance involves the description of the hull Ω_Λ as the inverse limit of approximants Γ_R that describe the point set Λ (or tiling) out to distance R around the origin. There is a natural map from $\mathbb{R}^d$ to Γ_R sending x to the equivalence class of $\Lambda - x$. A function or cochain is strongly pattern equivariant if it is the pullback of a function or cochain on a fixed approximant Γ_R.

Weakly pattern equivariant functions and cochains are defined as limits of strongly pattern equivariant objects. The precise definition depends on the category that we are working in.

- A continuous function f is weakly pattern equivariant (in the topological sense) if it can be approximated in the uniform topology by strongly pattern equivariant continuous functions, i.e. for all $\epsilon > 0$ there exists a strongly pattern equivariant continuous function h such that $\|f - h\|_\infty < \epsilon$.
- A cochain f on the tiling $\mathbf{T}$ derived from Λ is weakly pattern equivariant if it can be approximated in the sup-norm by strongly pattern equivariant cochains, i.e. for all $\epsilon > 0$ there exists a strongly pattern equivariant cochain h such that $\sup_c |f(c) - h(c)| < \epsilon$.

4.1. Dynamical eigenfunctions.

Pattern equivariant functions can be used to describe eigenfunctions, i.e. solutions to equations (1).

Lemma 4.1. *Let Λ be a repetitive FLC Delone set. β is a topological eigenvalue for Λ if and only if $e^{2\pi i\beta}$ is weakly pattern equivariant in the topological sense.*

Proof. It has been known for some time that weakly pattern equivariant functions in the topological sense are precisely the restrictions of continuous functions on Ω_Λ to the orbits through Λ.[1] Clearly, any continuous solution of (1) restricts to a function proportional to $e^{2\pi i\beta}$ on the orbit through Λ. $\square$

The following result is useful for determining whether an element $\beta \in \mathbb{R}^{d*}$ is a topological eigenvalue.

Lemma 4.2. *Let $\Lambda \in \mathbb{R}^d$ be an FLC Delone set and let $f : \mathbb{R}^d \to \mathbb{C}$ be a continuous function. f is weakly pattern equivariant if and only if it is uniformly continuous and for all $\epsilon > 0$ there exists $R > 0$ such that $(\Lambda - x) \cap B_R(0) = (\Lambda - y) \cap B_R(0)$ implies $|f(x) - f(y)| < \epsilon$.*

Proof. A weakly pattern equivariant continuous function extends to a continuous function on the hull of Λ. Since the hull is compact, this extension is uniformly continuous, implying that the original function is also uniformly continuous. To see the second property, approximate f to within $\frac{\epsilon}{2}$ by a strongly pattern equivariant continuous function.

For the converse, let f be a uniformly continuous function satisfying the second property of the lemma. Pick $\epsilon > 0$ and $R > 0$ such that $(\Lambda - x) \cap B_R(0) = (\Lambda - y) \cap B_R(0)$ implies $|f(x) - f(y)| < \epsilon$. Without loss of generality we may assume that f is real valued. Define

$$f_\epsilon(x) := \inf\{f(y) : (\Lambda - x) \cap B_R(0) = (\Lambda - y) \cap B_R(0)\}.$$

Then f_ϵ is strongly pattern equivariant and $|f_\epsilon(x) - f(x)| \leq \epsilon$. Moreover, the uniform continuity of f implies that f_ϵ is continuous. $\square$

The next two results concern functions F that are *a priori* only defined on Λ and therefore may be regarded as 0-cochains on the tiling $\mathbf{T}_\Lambda$. The coboundary δF of F is then a 1-cochain on $\mathbf{T}_\Lambda$.

Proposition 4.3. *Suppose that Λ is FLC and repetitive. Let $F : \Lambda \to \mathbb{R}$ be a bounded function such that the 1-cochain δF on $\mathbf{T}_\Lambda$ is strongly pattern equivariant. Then for all ϵ there exists a relatively dense set Λ_ϵ which is locally derivable from Λ such that $\forall p, q \in \Lambda_\epsilon$: $|F(p) - F(q)| < \epsilon$.*

Proof. Let $2M = \sup_{a,b\in\Lambda} F(b) - F(a)$. Then F takes values in $[f_0 - M, f_0 + M]$. Let $f = F - f_0$. Given $\epsilon > 0$ there are a, b such that $2M - \epsilon \leq f(b) - f(a) \leq 2M$. Hence $M - \epsilon \leq f(b) \leq M$ and $-M \leq f(a) \leq -M + \epsilon$. Let c be a chain with boundary $b - a$. Since δF is strongly pattern equivariant there exists a chain $\tilde{c}$ containing c such that the value $\delta F(c)$ depends only on $\tilde{c}$, in other words, whenever $\tilde{c} + t$ occurs in the complex then $\delta F(c + t) = \delta F(c)$. Let P be the set of t such that $\tilde{c} + t$ occurs in the complex. P is locally derived from Λ, and by repetitivity P is relatively dense. Taking $\Lambda_\epsilon = P + b$, we have $M - \epsilon \leq f(p) \leq M$ for all $p \in \Lambda_\epsilon$, and hence $|f(p) - f(q)| < \epsilon$ for all $p, q \in \Lambda_\epsilon$. Λ_ϵ is locally derived from Λ and relatively dense. $\square$

[1]This statement is e.g. implicit in [K1] in the proof that weakly pattern equivariant functions in the topological sense are isomorphic, as a C^*-algebra, to $C(\Omega_\Lambda)$.

Corollary 4.4. *Under the conditions of Prop. 4.3 F is weakly pattern equivariant.*

Note that boundedness of F is also a necessary criterion since any weakly pattern equivariant cochain is bounded.

Proof. Applying Prop. 4.3 to F we find that given any ϵ, there exists a relatively dense set Λ_ϵ that is locally derivable from Λ such that $\forall p, q \in P_\epsilon$: $|F(p) - F(q)| < \epsilon$. We partition Λ into subsets $\{V_a\}_a$ for $a \in \Lambda_\epsilon$ as follows. Let $\mathcal{V}(a)$ be the Voronoi domain of $a \in \Lambda_\epsilon$. In a generic situation, no point of Λ lies on the boundary of some $\mathcal{V}(a)$ and then we can define $V_a = \mathcal{V}(a) \cap \Lambda$. In a non-generic situation we can make choices to associate the points on a boundary to one of the Voronoi domains in a locally derivable way (e.g. by a directional criterion). As a result, we have obtained a partition of Λ that is locally derivable from Λ. Now define

$$F_\epsilon(a) := \inf\{F(b) : b \in \Lambda_\epsilon\}$$

for $a \in \Lambda_\epsilon$ and

$$F_\epsilon(x) = F_\epsilon(a) + \delta F(c)$$

for $x \in V_a$ where c is any 1-chain with boundary $x - a$. Since δF is strongly pattern equivariant and the Voronoi domains bounded we obtain that F_ϵ is strongly pattern equivariant. Furthermore, also $F(x) = F(a) + \delta F(c)$ so that $|F(x) - F_\epsilon(x)| \leq \epsilon$. □

5. Topological conjugacies

5.1. **Shape deformations and shape conjugacies.** Consider two FLC polyhedral tilings $\mathbf{T}$ and $\mathbf{T}'$. Each tiling is determined by its edges and by the location of a single vertex. Deforming the edges deformes the shapes and sizes of the tiles. $\mathbf{T}'$ is a (shape) deformation of $\mathbf{T}$ if $\mathbf{T}$ and $\mathbf{T}'$ have the same combinatorics, and if the vectors that give the displacements along each edge of $\mathbf{T}'$ is obtained in a local way from the corresponding edge of $\mathbf{T}$. That is, there exists $R > 0$ such that the displacement along edge e' of $\mathbf{T}'$, corresponding to edge e of $\mathbf{T}$, depends only on $B_R(e) \cap \mathbf{T}$. This does not mean that inspection of $B_R(e) \cap \mathbf{T}$ allows one to determine the location of e', only the relative position of the two endpoints. We can encode this by a vector-valued 1-form α on $\mathbf{T}$, namely, $\alpha(e) = v_{e'}$ where $v_{e'}$ is the displacement vector along edge e'. The requirement that $v_{e'}$ depends only on $B_R(e) \cap \mathbf{T}$ means that α is strongly pattern equivariant. Note that shape deformations automatically preserve finite local complexity.

There are two canonical ways to turn a shape deformation between two specific tilings $\mathbf{T}$, $\mathbf{T}'$ into a continuous surjection $\Omega_{\mathbf{T}} \to \Omega_{\mathbf{T}'}$. The first method preserves canonical transversality and applies to all shape deformations [SW, K2]. For simplicity we suppose that $\mathbf{T}$ has a vertex x_0 on the origin $0 \in \mathbb{R}^d$. Consider the vector-valued 0-cochain $\tilde{F}$ satisfying $\delta\tilde{F} = \alpha$ and $\tilde{F}(x_0) = 0$. Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a piecewise-linear extension[2] of $\tilde{F}$ to $\mathbb{R}^d$, and let

[2]Identifying the vertices of $\mathbf{T}$ with a subset of $\mathbb{R}^d$ we view $\tilde{F}$ as a map on this subset.

$\phi_1(\mathbf{T}-x) = \mathbf{T}' - h(x)$. This map is uniformly continuous on the orbit of $\mathbf{T}$, and so extends to a continuous surjection $\phi_1 : \Omega_{\mathbf{T}} \to \Omega_{\mathbf{T}'}$. Note that ϕ_1 does not commute with translations, precisely because the distances between vertices in $\mathbf{T}'$ is different from the distance between vertices in $\mathbf{T}$.

The second method only applies to very special shape deformations (namely those that are called asymptotically negligible in [CS2, K2]). Let $F = \tilde{F} - \mathrm{id}$. F is the vector-valued 0-cochain on $\mathbf{T}$ whose value at an arbitrary vertex $x \in \mathbf{T}$ is $x' - x$, where x' is the corresponding vertex of $\mathbf{T}'$. If F is weakly pattern equivariant, then F extends to a continuous map on $\Omega_{\mathbf{T}}$ implying that $\phi_2(\mathbf{T}-x) = \mathbf{T}' - x$ is also uniformly continuous. ϕ_2 thus extends to a continuous surjection $\phi_2 : \Omega_{\mathbf{T}} \to \Omega_{\mathbf{T}'}$. In fact, if $\mathbf{T}_1 \in \Omega_{\mathbf{T}}$, then $\phi_2(\mathbf{T}_1)$ is the tiling obtained by moving each vertex x of $\mathbf{T}_1$ by $F(x)$. By construction, this map commutes with translations and hence is a topological semi-conjugacy. We call this a *shape semi-conjugacy*, and if ϕ_2 is invertible a *shape conjugacy.* If F is strongly pattern equivariant, then the shape semi-conjugacy is a local derivation. Local derivations preserve the Meyer property, but we will see that general shape deformations do not.

The 0-cochain F is determined by the tilings $\mathbf{T}$ and $\mathbf{T}'$, but we can equally well construct shape semi-conjugacies directly from cochains. Let F be any weakly pattern equivariant vector-valued 0-cochain taking values that are less than half the separation of vertices in $\mathbf{T}$, such that δF is strongly pattern equivariant. Then we can construct $\mathbf{T}'$ from $\mathbf{T}$ by moving each vertex x of $\mathbf{T}$ by $F(x)$ and preserving the combinatorics. It follows from the above that there is a semi-conjugacy ϕ_2 from $\Omega_{\mathbf{T}}$ to $\Omega_{\mathbf{T}'}$. If F is small enough then this procedure can be inverted [K2] so that ϕ_2 is a shape conjugacy.

The following theorem implies Theorem 1.3 showing that shape conjugacies are essentially the only topological conjugacies that are not mutual local derivations.

Theorem 5.1. *Let Λ and Λ' be FLC Delone sets which are pointed topologically conjugate, i.e. there exists a topological conjugacy $\phi : \Omega_\Lambda \to \Omega_{\Lambda'}$ with $\phi(\Lambda) = \Lambda'$. For each $\epsilon > 0$ there exists a mutual local derivation ψ_ϵ and a shape conjugacy s_ϵ such that $\phi = s_\epsilon \circ \psi_\epsilon$ and s_ϵ moves the location of each point in each pattern by less than ϵ.*

Proof. Let R' be the minimum distance between distinct points of Λ', and pick $\epsilon < R'/3$. Since ϕ is uniformly continuous, there exists a δ such that any two tilings within distance δ map to tilings within ϵ of one another. Here we use the common distance between patterns: Λ_1 and Λ_2 have distance at most δ if their patterns in a ball of radius $1/\delta$ agree up to a translation of size at most δ. Thus by uniform continuity of ϕ there exists an $R = 1/\delta$ such that the locations of the points in Λ' in a ball of radius $1/\epsilon$ around $x \in \mathbb{R}^d$ are determined to within ϵ by the pattern of Λ in a ball of radius R around those points. In particular, each point in Λ' (say at location x_0) is associated with a pattern of radius R around x_0 in Λ. Translate these patterns by $-x_0$ to yield a set S of point patterns on the ball of radius

R around the origin. Note that if we have two such point patterns $P_{1,2}$ with $P_1 = P_2 - y$, then either $|y| < \epsilon$ or $|y| > R' - \epsilon$. (The latter can happen if P_1 determines the existence of several points of Λ'.) Since $R' - \epsilon$ is strictly greater than 2ϵ, there is an equivalence relation on S that two patterns are equivalent is they are translates by less than ϵ.

Since Λ is FLC, there are only finitely many equivalence classes in S. Pick representatives $\{P_1, \ldots, P_n\}$ of the equivalence classes. Define a point pattern Λ_ϵ in the following way: Λ_ϵ has a point at x if and only if one of the patterns $P_i + x$ appears in Λ. By construction Λ_ϵ is locally derivable from Λ. Moreover, the points of Λ_ϵ are in 1-1 correspondence with the points of Λ', and each point in Λ_ϵ is within ϵ of the corresponding point of Λ'. Let $\psi_\epsilon : \Omega_\Lambda \to \Omega_{\Lambda_\epsilon}$ be the local derivation defined by the above procedure.

We show that Λ_ϵ is actually MLD with Λ and ψ_ϵ is a mutual local derivation. In fact, since ϕ is a topological conjugacy, the pattern of Λ on a ball of radius R is determined, up to translation by ϵ, by the pattern of Λ' on a bigger ball, say of radius R''. This means that the pattern $P_i + x$ in Λ that generated a point $x \in \Lambda_\epsilon$ can be determined from a ball of radius $R'' + \epsilon$ around x in Λ'. Thus ψ_ϵ is injective. Since ψ_ϵ is an injective local derivation, ψ_ϵ is a mutual local derivation.

We claim that the map $s_\epsilon = \phi \circ \psi_\epsilon^{-1}$ is a shape conjucagy. Let F be the 0-cochain on $\mathbf{T}_{\Lambda_\epsilon}$ whose value on each point $x \in \Lambda_\epsilon$ is the displacement to the corresponding point in Λ'. By the above, Λ_ϵ and Λ are at most distance ϵ apart. So if ϵ is small enough then $\mathbf{T}_{\Lambda_\epsilon}$ and $\mathbf{T}_{\Lambda'}$ have the same combinatorics.[3] We show that the coboundary δF is strongly pattern equivariant: Since Λ_ϵ and Λ are MLD, a pattern in a quite large ball in Λ_ϵ determines the pattern in a large ball in Λ. We also saw that uniform continuity of ϕ implies that a large ball in Λ determines the pattern in Λ' up to a small overall translation. But overall translations don't matter for coboundaries and so the very large ball in Λ_ϵ determines δF exactly.

Since F is bounded and δF is strongly pattern equivariant, it follows from Cor. 4.4 that F is weakly pattern equivariant, and hence that s_ϵ is a shape conjugacy. □

In view of the preceding discussion we can also formulate the last theorem in the following way: Two FLC Delone sets Λ and Λ' are pointed topologically conjugate if and only if for each $\epsilon > 0$ there exists a FLC Delone set Λ_ϵ such that

(1) Λ and Λ_ϵ are mutually locally derivable,
(2) Λ' and Λ_ϵ are mutually asymptotically negligible shape deformations,
(3) within ϵ-distance of each point of Λ' lies a point of Λ_ϵ and vice versa.

6. Cantor bundles over tori with canonical $\mathbb{R}^d$-action

It has been known for quite some time that the hull of any FLC tiling is homeomorphic to a fiber bundle over the d-torus whose typical fiber is a compact totally disconnected space

[3]Exept in the situation that Λ has accidentally high symmetries. This case can be delt with by taking instead of $\mathbf{T}_{\Lambda'}$ the limit $\lim_{\epsilon\to 0} \mathbf{T}_{\Lambda_\epsilon}$ whose vertices are still Λ'.

such as a Cantor set [SW]. This means that there is a continuous surjection $X \xrightarrow{\pi} \mathbb{T}^d$ onto the d-torus such that the pre-images $\pi^{-1}(t)$, $t \in \mathbb{T}^d$, are all homeomorphic to a single compact totally disconnected space $\mathcal{F}$, the so-called typical fiber, and that X is locally a product i.e. every point has a neighbourhood U such that $\pi^{-1}(U)$ is homeomorphic to $\mathcal{F} \times U$. However, the construction of [SW] is based on deforming the tiling to a tiling by cubes. It yields a homeomorphism but in general not a topological conjugacy.

Let $X \xrightarrow{\pi} \mathbb{T}^d$ be any fiber bundle over the d-torus. We say that a (continuous) action of $\mathbb{R}^d$ on the bundle X is *canonical* if there exists a regular lattice $L \subset \mathbb{R}^d$ such that π becomes $\mathbb{R}^d$-equivariant (hence a factor map) when we identify $\mathbb{T}^d = \mathbb{R}^d/L$ and equip it with the action induced by translation on $\mathbb{R}^d$ (called a rotation action). Note that if the fiber of the bundle is totally disconnected then, once we have fixed the lattice L, the canonical action becomes unique.

Any fiber bundle over a d-torus with canonical $\mathbb{R}^d$-action has d independent topological eigenvalues. Indeed, the pull-back under π of any continuous eigenfunction of the rotation action on $\mathbb{T}^d$ is a continuous eigenfunction of $(X, \mathbb{R}^d)$. In particular the group of eigenvalues of $(\mathbb{T}^d, \mathbb{R}^d)$ (which is a regular lattice in $\mathbb{R}^{d*}$) is a subgroup of the group of eigenvalues of $(X, \mathbb{R}^d)$.

In the above context of fiber bundles over a d-tori with canonical $\mathbb{R}^d$-action we say that the action is *expansive* if the induced action on a fiber is expansive, that is, there exists a constant $\epsilon > 0$ such that for any $x, y \in \pi^{-1}([0])$, $\sup_{t \in L} d(t \cdot x, t \cdot y) \le \epsilon$ implies $x = y$. The largest such constant is called the expansivity constant. Here d is a metric which induces the topology. Whereas the expansivity constant will depend on the choice of d, the mere fact that the $\mathbb{R}^d$-action is expansive does not. There do exist Cantor bundles, such as the dyadic solenoid over the circle, with non-expansive canonical dynamics, but these are not homeomorphic to FLC tiling spaces.

If the $\mathbb{R}^d$ action on X is topologically transitive and $\pi : X \to \mathbb{T}^d$ is a factor map, then the action by rotation on the torus must be transitive as well. For dimensional reasons the action by rotation must then also be locally free. Since the action of $\mathbb{R}^d$ on X is then locally free as well, the fibers $\pi^{-1}([t])$ must be transversal to it. Furthermore the action of $t \in \mathbb{R}^d$ provides a homeomorphism between $\pi^{-1}([s])$ and $\pi^{-1}([s-t])$ and so all fibers are homeomorphic. Letting Ξ denote the fiber on $[0]$ we see that for small enough open balls $U \in \mathbb{T}^d$, $\Xi \times U \ni (\xi, [t]) \mapsto t \cdot \xi \in \pi^{-1}(U)$ (we lift the ball U to a ball in $\mathbb{R}^d$) is a homeomorphism providing a local trivialization for a bundle structure defined by π. Thus, once we have established that π is a factor map onto a rotation action, the only issue will be to show that some pre-image has the topological characterization we want and that the dynamics is expansive. We will do this in the next section.

7. Proof of Theorems 1.1 and 1.3

We prove the following three statements in turn. Taken together, they imply both Theorem 1.1 and Theorem 1.3.

(1) A repetitive Meyer set in $\mathbb{R}^d$ has a relative dense set of topological eigenvalues. In particular, it has d linearly independent topological eigenvalues.
(2) If an FLC Delone set has d linearly independent topological eigenvalues, then it is topologically conjugate to a bundle over the d torus with totally disconnected compact fiber and expansive canonical $\mathbb{R}^d$ action. If the Delone set is repetitive, then the $\mathbb{R}^d$ action is minimal.
(3) A bundle with totally disconnected compact fiber over the d torus with minimal, expansive canonical $\mathbb{R}^d$ action is topologically conjugate to the dynamical system of a repetitive Meyer set.

Proof of Statement 1. Let Λ be a Meyer set and pick $0 < \epsilon < 1$ and $a_0 \in \Lambda$. Using Lemma 4.1 we will establish that each $\beta \in \Delta^\epsilon$ is a continuous eigenvalue by showing that $f(x) := \exp(2\pi i\beta(x - a_0))$ is weakly pattern equivariant. Since Δ^ϵ is relatively dense this then proves Statement 1.

Since $|\exp(2\pi i\beta(a - a_0)) - 1| < \epsilon < 1$ for all $a \in \Lambda$, the real part of $f(a)$ is positive for all $a \in \Lambda$. There is then a function $\theta : \Lambda \to (-\pi/2, \pi/2)$ such that $f(a) = e^{i\theta(a)}$ for all $a \in \Lambda$. Consider the 1-cochain $\alpha = \delta\theta$ on T_Λ. It satisfies $-\pi < \alpha(c) < \pi$ for each edge c. We claim that $\alpha = \delta\theta$ is strongly pattern equivariant. Since $\exp(i\alpha)(c) = e^{i\theta(b)-i\theta(a)} = f(b - a)$ where a, b are the boundary vertices of c, we see that $\exp(i\alpha)$ is determined by the length and the direction of c and is thus strongly pattern equivariant. But since $-\pi < \alpha(c) < \pi$, α is itself strongly pattern equivariant. Hence $\delta\theta$ satisfies the conditions of Prop. 4.3. Hence for all $\eta > 0$ there exists Λ_η, locally derivable from Λ, such that $\forall a, b \in \Lambda_\eta$ we have $|\theta(a) - \theta(b)| < \eta$. Since $f(a) = e^{i\theta(a)}$ we conclude that for all $\eta > 0$ there exists Λ_η, locally derivable from Λ, such that $\forall a, b \in \Lambda_\eta$ we have $|f(a) - f(b)| < \eta$.

We claim that f satisfies the conditions of Lemma 4.2, which then proves Statement 1. Clearly f is uniformly continuous. Λ_η being locally derivable from Λ means that there exists a finite set $\{p_1, \cdots, p_k\}$ of patches p_i such that $\Lambda_\eta = \{x \in \Lambda : \exists i : p_i \subset \Lambda - x\}$. Let R be large enough so that each R-ball contains some patch p_i. Then, given x there are i and $t \in B_{R-r}(0)$ such that $p_i \subset (\Lambda - x - t) \cap B_R(0)$. In particular this means that $x + t \in \Lambda_\eta$. Hence if $(\Lambda - x) \cap B_R(0) = (\Lambda - y) \cap B_R(0)$ then $|f(x) - f(y)| = |f(x + t) - f(y + t)| < \eta$ as $x + t$ and $y + t$ belong to Λ_η. □

Note that we have proved considerably more than just the relative density of the topological eigenvalues. The topological eigenvalues form a group, so the set of topological eigenvalues contains the group generated by Δ^ϵ. In many (but not all) cases, this group is the same as the set of measurable eigenvalues. In these cases, all eigenvalues are topological!

This gives a new perspective on why the eigenfunctions for substitution tilings and for model sets can always be chosen continuous.

Proof of statement 2. The d topological eigenvalues β_i generate a lattice $L^* \subset \mathbb{R}^{d*}$. Let $L \subset \mathbb{R}^d$ be the dual (or reciprocal) lattice to L^*. The d eigenfunctions are periodic with period L, and their values give a map $\pi(\omega) = (\beta_1(\omega), \cdots, \beta_d(\omega)) \in \mathbb{R}^d/L$ from the hull Ω to the torus $\mathbb{T}^d = \mathbb{R}^d/L$. By (1) π is $\mathbb{R}^d$-equivariant if we consider the rotation action on $\mathbb{T}^d$. The hull is thus a bundle over the d-torus with canonical $\mathbb{R}^d$-action.

Let $\Xi = \pi^{-1}([0])$. It is compact as Ω is compact. We wish to show that it is totally disconnected. For that we only use that the hull of an FLC tiling is a matchbox manifold, i.e. there is a finite open covering $\{\tilde{U}_i\}_{i\in I}$ of Ω such that $\tilde{U}_i \cong C_i \times U_i$ where C_i are totally disconnected compact sets and U_i open balls of $\mathbb{R}^d$ and the $\mathbb{R}^d$-action is given locally by translation in the second coordinate: $t\cdot(c,u) = (c, u-t)$ provided $u-t \in U_i$. Let $\Xi_i := \Xi \cap \overline{\tilde{U}_i}$. We explained above that Ξ is transversal to the action. This means that return vectors to Ξ have a length which is bounded from below by some strictly positive number l. If we choose the sets U_i to be of diameter smaller than l then Ξ_i intersects the plaquettes $\{c\} \times \overline{U_i}$ at most once. Hence the projection $pr_1 : C_i \times \overline{U_i} \to C_i$ onto the first factor restricts to a continuous bijection between Ξ_i and its image $pr_1(\Xi_i)$. Since Ξ_i is compact this bijection is bi-continuous. Since $pr_1(\Xi_i)$ is a compact subset of a totally disconnected space it is itself totally disconnected. Thus Ξ_i is totally disconnected. Since finite unions of totally disconnected sets are totally disconnected, $\Xi = \bigcup_i \Xi_i$ is totally disconnected.

We next show expansivity. For this we choose the standard tiling metric on the hull Ω. By finite local complexity, if two tilings agree exactly at a spot and the nearest neighbor tiles do not agree exactly, then the nearest neighbor tiles either differ in label or differ in position by at least a fixed quantity ϵ. If two tilings $\mathbf{T}_{1,2} \in \Xi$, then either $d(\mathbf{T}_1, \mathbf{T}_2) \geq \epsilon/2$ or there is a translate $\mathbf{T}_1'$ of $\mathbf{T}_1$ by less than $\epsilon/2$ such that $\mathbf{T}_1'$ and $\mathbf{T}_2$ agree exactly at the origin. But then $\mathbf{T}_1'$ and $\mathbf{T}_2$ disagree by at least ϵ at the nearest tile where they don't agree exactly, so translates of $\mathbf{T}_1'$ and $\mathbf{T}_2$ differ by at least ϵ, so translates of $\mathbf{T}_1$ and $\mathbf{T}_2$ disagree by at least $\epsilon/2$. This proves that the $\mathbb{Z}^d$ action on Ξ is expansive with expansivity constant $\epsilon/2r$, where r is the diameter of a fundamental domain of L. The last statement is well known. □

It is not difficult to see that if the Delone set is repetitive and does not have d independent periods, then the fiber of the bundle has no isolated points and so is a Cantor set. Indeed, by minimality Ξ consists either of a single (necessarily finite) orbit, or has no isolated points. The first case arises precisely if the tiling is totally periodic, with d independent periods.

We remark that the expansivity of the action could also be concluded from the work of Benedetti-Gambaudo [BG] who define the concept of expansivity more generally for tiling dynamical systems on homogeneous spaces using solonoids.

Proof of Statement 3. Let Ξ be the totally disconnected compact fiber over $[0] \in \mathbb{T}^d$ (a point we can actually choose) of the bundle $\Omega \to \mathbb{T}^d$. The canonical expansive action induces an expansive action of $L \cong \mathbb{Z}^d$ on Ξ. Let ϵ be the expansivity radius. Pick a finite clopen cover of Ξ by sets of diameter less than ϵ, and label these sets with a finite alphabet $\mathcal{A} = \{1, \ldots, n\}$. To each element $\xi \in \Xi$ associate an element $u_\xi \in \mathcal{A}^{\mathbb{Z}^d}$, where $u_\xi(t)$ labels which clopen set contains $t \cdot \xi$. The map from $\Xi \to \mathcal{A}^{\mathbb{Z}^d}$ is injective, so we can view Ξ as a subshift of $\mathcal{A}^{\mathbb{Z}^d}$. This then makes Ω topologically conjugate to a tiling space where each tile is a decorated fundamental domain of the lattice L. The decoration is given by a letter of the alphabet $\mathcal{A}$ but each letter i may as well be encoded by a finite set A_i. Let then Λ be the set of vertices of that tiling together with the points of the sets which decorate the tiles. Then Λ is MLD with the tiling and furthermore $\Lambda \subset L + \bigcup_{i=1}^n A_i$ showing that it is Meyer.

Minimality of the bundle implies minimality of the dynamical system of Λ and hence its repetitivity. □

Combining Theorem 1.1 with Theorem 5.1 we obtain the following corollary.

Corollary 7.1. *Let Λ be an FLC repetitive Delone set with d independent topological eigenvalues. For each $\epsilon > 0$ there is a Meyer set Λ_ϵ such that* $\mathrm{dist}(\Lambda, \Lambda_\epsilon) \leq \epsilon$*, where* dist *denotes Hausdorff distance between sets of $\mathbb{R}^d$.*

This does not imply that Λ is itself Meyer, as the minimal distance between points of $\Lambda_\epsilon - \Lambda_\epsilon$ depends on ϵ.

8. Counterexamples

In this section we provide examples of point sets with unusual behavior that was suspected to be impossible. The first two examples are based on the scrambled Fibonacci sequence of [FS]. They are obtained by "scrambling" the Fibonacci substitution.

8.1. **Scrambled Fibonacci sequences.** Let $\sigma, P_1 : \mathcal{A} \to \mathcal{A}^*$ be two maps on the alphabet $\mathcal{A}$. Both maps are extended in the usual way as morphisms on $\mathcal{A}^*$. Let $P_2 = P_1 \circ \sigma$. Then one obtains $P_2(a)$ upon replacing each letter o in $\sigma(a)$ by the word $P_1(o)$. We may also think of σ as a rule to compose (or fuse) words of the first generation $\{P_1(o), o \in \mathcal{A}\}$ to form words of the next generation $P_2(a)$.

Now suppose that we have a family of substitutions $\{\sigma^{(n)}\}_{n \in \mathbb{N}}$ and let $P_0 = \mathrm{id}$. The composition $P_n = P_{n-1} \circ \sigma^{(n)}$ then describes the fusion of words of generation $n-1$ to words of generation n. This is an example of a fusion rule [FS].[4] The sequence space associated to the fusion rule is the subset X of $\mathcal{A}^{\mathbb{Z}}$ of doubly infinite sequences whose factors are sub-words of some $P_n(o)$, $o \in \mathcal{A}$, $n \in \mathbb{N}$. Words of the form $P_n(o)$ are called nth order superletters.

[4] Even in one dimension this is not the most general type of fusion rule, but it is the one we need.

If all $\sigma^{(n)}$ are equal to a fixed substitution σ then X is the substitution sequence space associated to σ.

For concreteness we will consider the Fibonacci substitution σ on $\mathcal{A} = \{a, b\}$ defined by $\sigma(a) = ab$, $\sigma(b) = a$ and denote $F_n = P_n$, i.e. $F_n = \sigma^n$. We denote its sequence space X by X_F. Recall that $F_n(a)$ contains f_{n+1} letters a and f_n letters b while $F_n(b)$ contains f_n letters a and f_{n-1} letters b. Here f_n is the nth Fibonacci number defined iteratively by $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$.

Let $N(n)$ be a strictly and fast increasing function on $\mathbb{N}$ and $\Delta(n) = N(n) - N(n-1)$. Let $A_n = F_{N(n)}$, which we may also write as $A_n = A_{n-1} \circ \sigma^{\Delta(n)}$, viewing $A_n(o)$ as the nth order superletters to the fusion rule $\{\sigma^{\Delta(n)}\}_{n\in\mathbb{N}}$. The sequence space X_F can now also be described as the subset of doubly infinite sequences in $\mathcal{A}^{\mathbb{Z}}$ whose factors are sub-words of some $A_n(a)$, $a \in \mathcal{A}$, $n \in \mathbb{N}$.

Extend the alphabet with an extra letter $\tilde{\mathcal{A}} = \mathcal{A} \cup \{e\}$. Let κ denote an arbitrary odd natural number. Define the fusion rule by the following family, $\tilde{\sigma}^{(n)} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}^*$

$$\tilde{\sigma}^{(n)}(x) = \begin{cases} \sigma^{\Delta(n)}(x) & \text{if } x \in \mathcal{A},\ n \text{ odd} \\ \sigma_e^{\Delta(n)}(x) & \text{if } x \in \mathcal{A},\ n \text{ even} \\ \sigma_r^{\Delta(n)}(b) & \text{if } x = e \end{cases}$$

where $\sigma_e^{\Delta(n)}(x)$ is the word obtained from $\sigma^{\Delta(n)}(x)$ by replacing the last letter b with e, and $\sigma_r^{\Delta(n)}(b)$ is the word obtained from $\sigma^{\Delta(n)}(b)$ by rearranging all the a's in front of the b's, that is $\sigma_r^{\Delta(n)}(b) = a^{f_{\Delta(n)}} b^{f_{\Delta(n)-1}}$. We denote by $S_n(x)$ the nth order superletters of this fusion rule. That is, for κ odd,

$$S_\kappa(o) = S_{\kappa-1} \circ \sigma^{\Delta(\kappa)}(o), \text{ for } o \in \mathcal{A}, \quad S_\kappa(e) = S_{\kappa-1} \circ \sigma_r^{\Delta(\kappa)}(b) \tag{2}$$

$$S_{\kappa+1}(o) = S_\kappa \circ \sigma_e^{\Delta(\kappa+1)}(o), \text{ for } o \in \mathcal{A}, \quad S_{\kappa+1}(e) = S_\kappa \circ \sigma_r^{\Delta(\kappa+1)}(b) \tag{3}$$

The subset X_S of doubly infinite sequences in $\tilde{\mathcal{A}}^{\mathbb{Z}}$ whose factors are sub-words of some $S_n(x)$, $n \in \mathbb{N}$ is by definition the space of *scrambled Fibonacci sequences*. Note that none of the 1-superletters contain the letter e, so this is in fact a subset of $\mathcal{A}^{\mathbb{Z}}$. In fact, none of the odd-order superletters ever contain an even-order superletter of type e, so the definition of $S_{\kappa+1}(e)$ is in fact irrelevant.

In other words, the scrambling consists of two steps. At each odd level we introduce a germ $S_\kappa(e)$ made from two periodic pieces. The presence of $S_\kappa(e)$ superletters with κ large will serve to eliminate topological eigenvalues. At even levels we insert a single $S_k(e)$ into $S_{k+1}(b)$. As long as $\Delta(\kappa + 1)$ grows quickly enough, this makes the e-superletters too rare to affect the measurable dynamics, and in fact tiling spaces based on X_S will prove to be measurable conjugate to those based on X_F.

We say that a fusion rule $\{S_n\}$ is recognizable if for each n, each sequence in X_S can be uniquely decomposed into a concatenation of sequences of superletters $S_n(x)$. For substitution tilings, recognizability is an automatic consequence of non-periodicity. For fusions, it must be checked separately.

Proposition 8.1 ([FS]). *The scrambled Fibonacci fusion rule is recognizable.*

Proof. We proof this by induction. Let $\xi \in X_S$. Let κ be odd and assume that ξ can uniquely be written as a sequence of $(\kappa-1)$-superletters (this is trivially true for $\kappa = 1$). It is easy to see that $\sigma^{\Delta(\kappa)}(o)$ starts with a and does not contain a b^2. Thus we can identify the powers $S_{\kappa-1}(b)^{f_{\Delta(\kappa)}-1}$ in ξ and regroup $S_{\kappa-1}(a)^{f_{\Delta(\kappa)}}S_{\kappa-1}(b)^{f_{\Delta(\kappa)}-1}$ to $S_\kappa(e)$. The remaining $S_{\kappa-1}(a)$'s and $S_{\kappa-1}(b)$'s can be regrouped by applying $\Delta(\kappa)$ times the procedure known from inverting the Fibonacci substitution, namely first regroup all words ab to a and then replace the remaining a's (not the new ones!) by b. As for the Fibonacci substitution one sees that this is the only way to regroup.

We now show that we can group the sequence into $(\kappa+1)$-superletters. Having grouped the sequence into κ-superletters, one can identify the $S_{\kappa+1}(b)$'s by the presence of an $S_\kappa(e)$. The remaining $\kappa+1$-supertiles are all of type a, since even-order superletters of type e do not appear. $\square$

Let $\xi^{(n)}$ denote the nth order superletter that contains $\xi(0)$. (By recognizability, this is uniquely defined.) Let $X_S^* \subset X_S$ be the set of all sequences ξ such that (a) the union of the words $\xi^{(n)}$ is all of ξ, and not just a half-line, and (b) only finitely many $\xi^{(n)}$'s are of type e. For each $\xi \in X_S^*$, let n_0 be the largest value of n for which $\xi^{(n)}$ is of type e (or zero if none of these superletters are of type e.)

We may define a shift equivariant map

$$\psi : X_S^* \to X_F$$

in the following way: $\psi(\xi) = \lim_{n\to\infty}\psi_n(\xi)$ where, for $n \geq n_0$, $\psi_n(\xi)$ is obtained from ξ by replacing $\xi^{(n)}$ with the corresponding Fibonacci superletter $A_n(o)$, where $\xi^{(n)}$ is of type o. Note that $\psi_{n+1}(\xi)$ and $\psi_n(\xi)$ agree on the image of $\xi^{(n)}$, since neither $\xi^{(n)}$ nor $\xi^{(n+1)}$ is of type e. The limit is taken in the product topology. Shift-equivariance is guaranteed by the fact that the union of the $\xi^{(n)}$'s is all of ξ.

We cannot expect ψ to be continuous, nor do we have a reasonable extension of ψ to all of X_S. However, ψ is measurable. With appropriate conditions on $N(n)$, X_S^* has full measure with respect to the (unique [FS]) invariant Borel probability measure on X_S. In fact, the frequency of the superletter $S_\kappa(e)$ in any sequence $\xi \in X_S$ is bounded by $\phi^{-\Delta(\kappa+1)}$, since there is only one $S_\kappa(e)$ per $S_{\kappa+1}(b)$. As a result:

Lemma 8.2 ([FS]). *Let μ be an ergodic shift invariant Borel probability measure on X_S. If $\sum_{k\ odd}\phi^{-\Delta(k+1)} < \infty$ then X_S^* has full μ-measure.*

Corollary 8.3 ([FS]). *Under the assumptions of the last lemma, ψ defines a conjugacy between the shift measure dynamical systems $(X_S, \mu, \mathbb{Z})$ and $(X_F, \mathbb{Z})$ equipped with the unique ergodic Borel probability measure. In particular, $(X_S, \mathbb{Z})$ is uniquely ergodic.*

We recall also from [FS] that the shift dynamical system $(X_S, \mathbb{Z})$ is minimal.

8.2. **Scrambled Fibonacci tilings.** We now make tilings out of sequences by suspending the letters to intervals which we call tiles. We give the tile corresponding to a length L and the tile corresponding to b length S. Images of superletters are called supertiles. We will denote the corresponding tiling spaces by $\Omega_F^{L,S}$ and $\Omega_S^{L,S}$. As for the discrete systems on obtains that if $\sum_{\kappa\ odd} \phi^{-\Delta(\kappa+1)} < \infty$ then $(\Omega_S^{L,S}, \mathbb{R})$ is uniquely ergodic and measure conjugate to $(\Omega_F^{L,S}, \mathbb{R})$ so that in particular their dynamical spectra coincide. Furthermore, if $L\phi + S = L'\phi + S'$ then deformation theory shows that $(\Omega_F^{L,S}, \mathbb{R})$ is topologically conjugate to $(\Omega_F^{L',S'}, \mathbb{R})$ [RS]. We are asking which eigenvalues of $(\Omega_S^{L,S}, \mathbb{R})$ are topological and the answer will depend on the values of L, S. To answer this question we make use of the following corollary to Lemma 4.2. By a return vector between n-supertiles we mean a vector v such that in some high order supertile we find the n-supertile $S_n(o)$ at position x and at position $x+v$. For a real number r let $\|r\|$ denotes the distance from r to the nearest integer.

Corollary 8.4. *β is a topological eigenvalue if and only if for all $\epsilon > 0$ there exists an n_0 such that for all return vectors v between supertiles of order $n \geq n_0$ one has $\|\beta(v)\| \leq \epsilon$.*

This corollary applies to the Fibonacci tiling and to the scrambled Fibonacci tilings for all values of L, S. For the Fibonacci tiling with $L = \phi$ and $S = 1$ it is known that all eigenvalues are topological and then one can read also from the corollary that a necessary and sufficient condition for $\beta \in \mathbb{R}^* = \mathbb{R}$ to be an eigenvalue is $\|\beta\phi^n\| \to 0$ which is equivalent to $\beta \in \frac{1}{\sqrt{5}}\mathbb{Z}[\phi]$. After rescaling the tiles by $\sqrt{5}$ and doing a shape change that preserves $\phi L + S$, we get that the group of eigenvalues for $(\Omega_F^{1,1}, \mathbb{Z})$ is $\mathbb{Z}[\phi]$.

Theorem 8.5 ([FS]). *Suppose that $\Delta(\kappa) \geq N(\kappa-1)$ for odd κ and that $\sum_{\kappa\ odd} \phi^{-\Delta(\kappa+1)} < \infty$. If $L = \phi$ and $S = 1$ then the only topological eigenvalue is 0.*

Proof. Suppose that $\beta \in \mathbb{R}^* = \mathbb{R}$ is a topological eigenvalue. By Corollary 8.4, the maximum value of $\|\beta v\|$, where v is a return vector to $\kappa-1$-supertiles of type a, must go to zero as $\kappa \to \infty$. Likewise, $\|5\beta v\| \leq 5\|\beta v\|$ must go to zero. Note that the scrambling was made in such a way that $S_n(a)$ has the same length as $F_{N(n)}(a)$, namely $\phi^{N(n)+1}$. Since $S_\kappa(e)$ contains $f_{\Delta(\kappa)}$ consecutive $S_{\kappa-1}(a)$'s, and since $\Delta(\kappa) \geq N(\kappa-1)$, v can take on the particular values $v_1 = f_{N(\kappa-1)-1}|S_{\kappa-1}(a)| = f_{N(\kappa-1)-1}\phi^{N(\kappa-1)+1}$ and $v_2 = f_{N(\kappa-1)-2}|S_{\kappa-1}(a)| = f_{N(\kappa-1)-2}\phi^{N(\kappa-1)+1}$. Using the identity $f_n = (\phi^n - (-\phi)^{-n})/\sqrt{5} = (\phi^n - (-\phi)^{-n})(\phi + \phi^{-1})/5$, we obtain

$$\|5\beta v_m\| = \|\beta(\phi^2 + 1)(\phi^{2N(\kappa-1)-m} - (-1)^{N(\kappa-1)-m}\phi^m)\|,$$

where $m = 1$ or 2. Since β is a measurable eigenvalue, $\|\beta\phi^n\| \to 0$ as $n \to \infty$. This means that $\|\beta v_m\|$ can only go to zero if $\beta(\phi^2+1)\phi^m \in \mathbb{Z}$. However, ϕ is irrational, so this cannot be true for both $m = 1$ and $m = 2$ unless $\beta = 0$. □

Corollary 8.6. *Suppose that $\Delta(\kappa) \geq N(\kappa-1)$ for odd κ and that $\sum_{\kappa \text{ odd}} \phi^{-\Delta(\kappa+1)} < \infty$. Then the scrambled Fibonacci tiling with lengths $|a| = \phi$ and $|b| = 1$ is repetitive and pure point diffractive but not Meyer.*

Theorem 8.7. *Suppose that $\Delta(\kappa) \geq N(\kappa-1)$ for κ odd and that $\sum_{\kappa \text{ odd}} \phi^{-\Delta(\kappa+1)} < \infty$. If $L = S = 1$ then the group of topological eigenvalues is $\mathbb{Z}$.*

Proof. Since $L = S = 1$ all return vectors are integers and hence any integer β is a topological eigenvalue. To prove that the rest of the dynamical spectrum is not topological we use the same idea as for the preceding theorem. The difference is that now

$$|S_{\kappa-1}(a)| = f_{N(\kappa-1)+1} + f_{N(\kappa-1)} = \frac{1}{\sqrt{5}}(\phi^{N(\kappa-1)+2} - (-\phi)^{-N(\kappa-1)-2}).$$

Accordingly we obtain as necessary criterion for β to be a topological eigenvalue that

$$\|5\beta v_m\| = \|\beta(\phi^{N(\kappa-1)-m} - (-\phi)^{-N(\kappa-1)+m})(\phi^{N(\kappa-1)+2} - (-\phi)^{-N(\kappa-1)-2})\| \stackrel{\kappa\to+\infty}{\longrightarrow} 0$$

for $m = 1$ or 2. Since $\|\beta\phi^n\| \to 0$ as $n \to \pm\infty$, we conclude that $\beta(\phi^{-m-2} + (-1)^m\phi^{m+2}) \in \mathbb{Z}$ for $m = 1$ or $m = 2$. Since $\beta(\phi^{-3} - \phi^3) = -4\beta$ and $\beta(\phi^{-4} + \phi^4) = 7\beta$ are integers, $\beta \in \mathbb{Z}$. □

Corollary 8.8. *Suppose that $\Delta(\kappa) \geq N(\kappa-1)$ for odd κ and that $\sum_{\kappa \text{ odd}} \phi^{-\Delta(\kappa+1)} < \infty$. Then the scrambled Fibonacci tiling with equal tile lengths is a repetitive pure point diffractive Meyer tiling having some eigenvalues that are not topological. The topological eigenvalues form a direct summand subgroup of rank 1 in the full group of eigenvalues which has rank 2.*

8.3. **A non-Meyer shape change of a Meyer tiling.**

Theorem 8.9. *There are FLC sets that are topologically conjugate to Meyer sets but are not themselves Meyer.*

Proof. Consider the irreducible substitution σ on three letters, with $\sigma(a) = abca$, $\sigma(b) = abb$, and $\sigma(c) = ac$. Since each substituted letter begins with a, the first cohomology of the resulting tiling space is the direct limit of the transpose of the subsitution matrix $M = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ [BD]. The eigenvalues of this matrix are $\lambda_1 \approx 3.247$, $\lambda_2 \approx 1.555$, and $\lambda_3 \approx .1981$. Call the eigenvectors ξ_1, ξ_2 and ξ_3. Since the third eigenvalue is less than 1 in magnitude, the corresponding eigenvector is represented by a weakly exact cochain that describes a shape conjugacy [CS2]. The first and second eigenvalues are greater than 1, and no combination of the two can be represented by a weakly exact cochain.

Pick a bi-infinite sequence coming from the substitution, and consider two tilings corresponding to that sequence. In one, all of the tiles have length 1. In the other, the tile lengths are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ plus a small multiple of the third eigenvector, chosen so that the resulting tile lengths are not rationally related. The left endpoints of the a tiles give us point sets Λ and Λ'.

Since Λ is a subset of the integer lattice, it is Meyer. Λ' is topologically conjugate to Λ. We will show that Λ' is not Meyer.

For each finite word w in our space of sequences, let $\ell(w)$ be the "population vector" in $\mathbb{Z}^3$ that counts the number of a's, b's and c's in w. Consider the possible population vectors of words of length n. The deviation from the average population for a word of length n is governed by eigenvalues λ_2 and λ_3. Since $|\lambda_3| < 1$, there is an upper bound to the inner product of $\ell(w)$ with ξ_3. However, since $\lambda_2 > 1$, the range of possible inner products with ξ_2 increases with n, going as $n^{\ln(\lambda_2)/\ln(\lambda_1)}$. Thus the number of possible population vectors for a given length n is bounded above and below by constants times $n^{\ln(\lambda_2)/\ln(\lambda_1)}$. [5] Since the lengths of the different tiles are irrationally related, each population vector gives a different spacing (in the tiling) between letters that are n apart (in the sequence). This means that the number of spacings between letters that are at most n apart grows faster than n. By the pigeonhole principle, the set of possible spacings of letters cannot be uniformly discrete, so the set of all tile vertices is not Meyer.

The same argument applies to substituted letters, implying that the left endpoints of the 1-supertiles are not a Meyer set. However, every 1-supertile begins with an "a" tile, so the left endpoints of the 1-supertiles are a subset of Λ'. Since a subset of a Meyer set is Meyer, Λ' cannot be Meyer. $\square$

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